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Mean eigenvalues for simple, simply connected, compact Lie groups

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Abstract

We determine for each of the simple, simply connected and compact Lie groups $SU(n)$, $Spin(4n+2)$ and E_6 with a complex fundamental representation that particular region inside the unit disc in the complex plane which is filled by their mean eigenvalues. We give analytical parameterizations for the boundary curves of these so-called trace figures. The area enclosed by a trace figure turns out to be a rational multiple of π in each case. We calculate also the length of the boundary curve and determine the radius of the largest circle that fits into a trace figure. The discrete centre of the corresponding compact Lie group shows up prominently in the form of cusp points of the trace figure placed symmetrically on the unit circle. For the exceptional Lie groups G_2 , F_4 and E_8 with trivial centre we determine the (negative) lower bounds on their mean eigenvalues lying within the real interval $[-1, 1]$. We find the rational boundary values $-2/7$, $-3/13$ and $-1/31$ for G_2 , F_4 and E_8 , respectively.

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1. Introduction and summary

The dynamical understanding of the confinement–deconfinement phase transition as a function of the temperature in a non-Abelian gauge theory, such as quantum chromodynamics, is a topic of current interest. The so-called Polyakov loop, given by the trace of the thermal Wilson line, has been proposed as an order parameter for the confinement–deconfinement transition [1]. Numerical simulations of pure $SU(3)$ Yang–Mills gauge theory on an Euclidian spacetime lattice have verified this proposal by computing the renormalized Polyakov loop as a function of the temperature [2] (see figure 4 therein). Because of its special role as an order parameter of the confinement–deconfinement transition effective Lagrangian in the Polyakov loop variable have been formulated [3]. There is also ongoing activity to interpret within such a framework the results of lattice QCD simulations at finite temperature including dynamical quarks [4, 5].

From the mathematical point of view the Polyakov loop variable is a complex number given by $1/3$ of the trace of a special unitary 3×3 matrix (i.e., $\frac{1}{3} \operatorname{tr} U$, $U \in SU(3)$). According to this definition as an arithmetic mean of three unitary eigenvalues (with product equal to 1) only restricted complex values are possible for the Polyakov loop variable, and this feature should be respected in the construction of any effective Lagrangian. We will show here in section 2 that the allowed region of $\frac{1}{3} \operatorname{tr} U$, $U \in SU(3)$ lies inside the unit circle and is bounded by a quartic curve with three cusp points at the cube roots of unity: 1 , $(-1 \pm i\sqrt{3})/2$. Further geometrical properties of this so-called trace figure of $SU(3)$, such as its enclosed area and the length of its circumference will also be derived.

The question about the domain of their mean eigenvalues naturally generalizes to the other compact Lie groups with a complex fundamental representation. The simple and simply connected, compact Lie groups (without any Abelian $U(1)$ -factors) are particularly interesting, since for these the trace figure (i.e., locus of the mean eigenvalues or image of the normalized fundamental character) will not be the entire unit disc in the complex plane. In section 3 we analyse the special unitary groups $SU(n)$, $n \geq 4$ and show that their trace figures are bounded by hypocycloids with cusp points at the n th roots of unity. Section 4 deals then with the spin groups $\operatorname{Spin}(4n+2)$ (the two-sheeted covering groups of the special orthogonal groups $SO(4n+2)$) which possess a complex fundamental representation of dimension 4^n . The exceptional Lie group E_6 with its complex fundamental representation of dimension 27 will be analysed in section 5. For the compact Lie groups G with only selfconjugate representations the mean eigenvalues are necessarily real and restricted to the interval $[-1, 1]$. If furthermore the centre of the group $Z(G)$ contains a factor \mathbb{Z}_2 , it follows by continuity that the whole interval $[-1, 1]$ gets filled by the mean eigenvalues of G . This leaves for the exceptional Lie groups G_2 , F_4 and E_8 with trivial centre the question about a lower bound on their mean eigenvalues. We will determine in section 6 these lower bounds for G_2 , F_4 and E_8 as the negative rational numbers $-2/7$, $-3/13$ and $-1/31$, respectively.

2. Special unitary group $SU(3)$

The simplest among the compact Lie groups with a complex fundamental representation is the special unitary group $SU(3)$, with numerous applications in theoretical physics. The question about the domain in the complex plane filled by the mean eigenvalues of $SU(3)$ -matrices is readily answered. The complex numbers of the form:

$$X + iY = \frac{1}{3} \operatorname{tr} U, \quad U \in SU(3), \quad (1)$$

as given by the normalized fundamental character can be rewritten in terms of the three unitary eigenvalues $z_j = e^{i\theta_j} \in U(1)$ of U , which are subject to the constraint $z_1 z_2 z_3 = 1$, as

$$X + iY = \frac{1}{3} \left(z_1 + z_2 + \frac{1}{z_1 z_2} \right). \quad (2)$$

Evidently, any such complex number $X + iY$ lies inside the unit disc $X^2 + Y^2 \leq 1$. In order to determine the extremal boundary values we eliminate one of the two angles by imposing the zero-derivative condition: $\partial(X + iY)/\partial z_1 = 0$, which gives $z_2 = z_1^{-2}$. Reinserting this relation leads already to the following parameterization of the boundary curve:

$$3(X_b + iY_b) = 2z_1 + z_1^{-2} = 2 \cos^2 \theta + 2 \cos \theta - 1 + 2i(1 - \cos \theta) \sin \theta, \quad (3)$$

where θ is an angle running from 0 to 2π . The grey-shaded area in figure 1 corresponds to the region in the complex plane which gets filled by the mean eigenvalues of $SU(3)$ -matrices. Besides a dihedral symmetry D_3 one observes cusps at the cube roots of unity 1 , $(-1 \pm i\sqrt{3})/2$

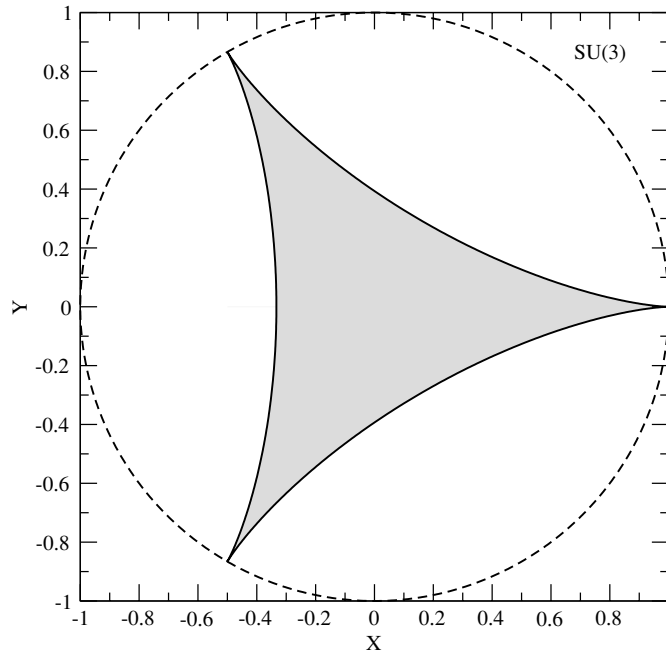


Figure 1. Region in the complex plane filled by the mean eigenvalues of $SU(3)$ -matrices. The area inside the (grey shaded) trace figure of $SU(3)$ is $2\pi/9$. The boundary curve is a quartic.

which obviously correspond to the centre of the compact Lie group $SU(3) : Z(SU(3)) = \mathbb{Z}_3$. We can also give a purely algebraic description of the boundary curve in figure 1. Eliminating $\cos \theta$ from the real part of equation (3) and inserting the solution of that quadratic equation into the squared imaginary part gives

$$Y_b^2 = \pm \frac{2}{\sqrt{3}}(1 + 2X_b)^{3/2} - 1 - 4X_b - X_b^2, \tag{4}$$

where the signs \pm correspond to the two branches for $-1/2 \leq X_b \leq -1/3$ in both the upper and the lower halfplane. After some further elementary transformations we arrive at the result that the trace figure of $SU(3)$ (i.e., the locus of its mean eigenvalues) is the region:

$$(1 + 3X)(1 - X)^3 - 6Y^2(1 + 4X + X^2) - 3Y^4 \geq 0, \tag{5}$$

inside the unit disc $X^2 + Y^2 \leq 1$ which is bounded by a *quartic* curve. With the help of the parameterization in equation (3) we can now compute several other geometrically interesting properties of the trace figure of $SU(3)$. The enclosed area amounts to

$$\Omega_3 = -2 \int_0^\pi d\theta \frac{dX_b}{d\theta} Y_b = \frac{2\pi}{9}, \tag{6}$$

and the length of its boundary curve is

$$L_3 = 6 \int_0^{\pi/3} d\theta \sqrt{\left(\frac{dX_b}{d\theta}\right)^2 + \left(\frac{dY_b}{d\theta}\right)^2} = \frac{16}{3}. \tag{7}$$

The radius of the largest circle that fits into the trace figure, determined by the minimum of $X_b^2 + Y_b^2 = (5 + 4 \cos 3\theta)/9$, is easily found to be

$$R_3 = \frac{1}{3}. \tag{8}$$

For the Lie group $SU(3)$ of rank 2, one has the special situation that the complex number $Z = X + iY = \frac{1}{3} \operatorname{tr} U$ (subject to the constraint equation (5)) represents uniquely a conjugacy class of $SU(3)$. Therefore the normalized invariant integral for class functions can be converted into a two-dimensional integral over the trace figure:

$$\int_{SU(3)} dU f_{\text{cl}}(U) = \frac{27\sqrt{3}}{2\pi^2} \int_{\triangleright} d^2Z \sqrt{4(1 + Z^3 + Z^{*3}) - 3(1 + Z^*Z)^2} \tilde{f}(Z, Z^*). \quad (9)$$

The square-root type weighting function originates from the Weyl determinant [6] expressed in terms of the coordinates (X, Y) and \triangleright is the region described in equation (5), where the radicand is positive. The measure is $d^2Z = dX dY$.

3. Special unitary groups $SU(n), n \geq 4$

The previous considerations can be straightforwardly generalized to the higher special unitary groups $SU(n), n \geq 4$. The complex numbers:

$$X + iY = \frac{1}{n} \operatorname{tr} U, \quad U \in SU(n), \quad (10)$$

can be rewritten in terms of the $n - 1$ (independent) unitary eigenvalues z_j as

$$X + iY = \frac{1}{n} \left(z_1 + \dots + z_{n-1} + \frac{1}{z_1 \cdots z_{n-1}} \right). \quad (11)$$

For an extremal boundary point we eliminate all but one angle through the zero-derivative conditions: $\partial(X + iY)/\partial z_j = 0, j = 1, \dots, n - 2$ which give $z_1 \cdots z_{n-1} = z_1^{-1} = \dots = z_{n-2}^{-1}$. This implies $z_{n-1} = z_1^{1-n}$, and with that we obtain the following parameterization of the boundary curve in terms of $z_1 = e^{i\theta}$:

$$\begin{aligned} n(X_b + iY_b) &= (n - 1)z_1 + z_1^{1-n} \\ &= (n - 1) \cos \theta + \cos(n - 1)\theta + i[(n - 1) \sin \theta - \sin(n - 1)\theta]. \end{aligned} \quad (12)$$

The grey-shaded areas in figures 2, 3 and 4 correspond to the regions in the complex plane filled by the mean eigenvalues of $SU(4), SU(5)$ and $SU(6)$ -matrices, respectively. In each case one observes in addition to a dihedral symmetry D_n cusp points at the n th roots of unity which reflect the discrete centre subgroup $Z(SU(n)) = \mathbb{Z}_n$. The area enclosed by the trace figure of $SU(n)$ comes out as a rational multiple of π , namely,

$$\Omega_n = \frac{\pi}{n^2} (n - 1)(n - 2), \quad (13)$$

and the circumference of its boundary curve, parameterized in equation (12), is

$$L_n = \frac{8}{n} (n - 1). \quad (14)$$

The radius of the largest circle that fits into the trace figure of $SU(n)$, determined by minimum of $X_b^2 + Y_b^2 = [n^2 - 2n + 2 + 2(n - 1) \cos(n\theta)]/n^2$, is also readily found to be

$$R_n = 1 - \frac{2}{n}. \quad (15)$$

When extrapolating to large dimensions, $n \rightarrow \infty$, one realizes that the trace figure tends to fill out the whole unit disc, $\Omega_\infty = \pi$, whereas its circumference becomes significantly longer than the unit circle, $L_\infty = 8 > 2\pi$. This difference arises from the n oscillations of the boundary curve between $R_{\max} = 1$ and $R_{\min} = 1 - 2/n$. In the special case of $SU(4)$ we can furthermore

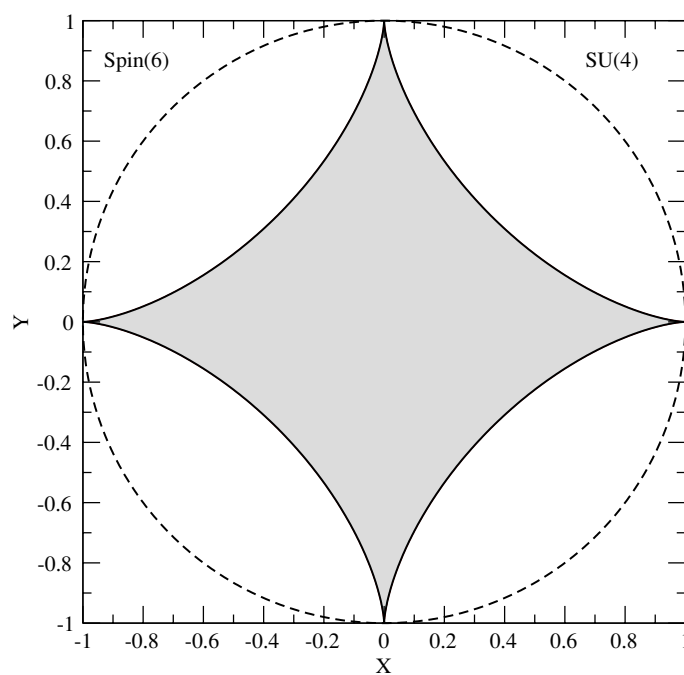


Figure 2. The trace figure of the Lie group $SU(4) = Spin(6)$. The enclosed area is $3\pi/8$. The boundary curve is a hypocycloid.

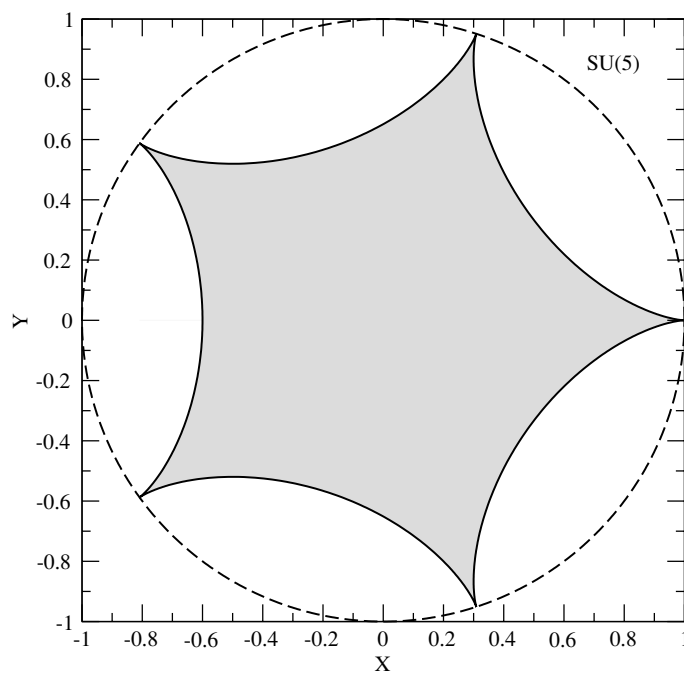


Figure 3. The trace figure of the Lie group $SU(5)$. The enclosed area is $12\pi/25$. The boundary curve is a hypocycloid.

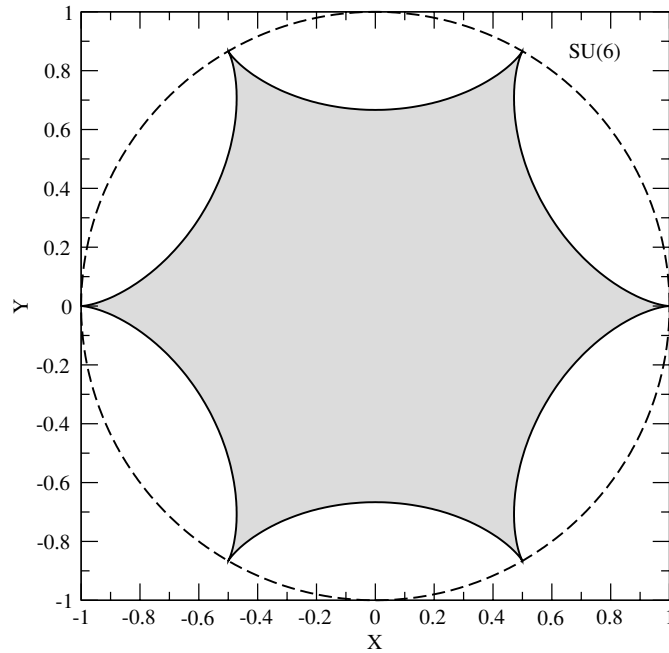


Figure 4. The trace figure of the Lie group $SU(6)$. The enclosed area is $5\pi/9$. The boundary curve is a hypocycloid.

employ the trigonometric identities: $3 \cos \theta + \cos 3\theta = 4 \cos^3 \theta$ and $3 \sin \theta - \sin 3\theta = 4 \sin^3 \theta$, and get the algebraic characterization:

$$|X|^{2/3} + |Y|^{2/3} \leq 1, \quad (16)$$

of the region filled by the mean eigenvalues of $SU(4)$ -matrices. Finally, it is interesting to note that the curves parameterized in equation (12) are the so-called hypocycloids. These hypocycloids are constructed by unrolling a circle of radius $1/n$ inside the unit circle and recording the motion of a point on the boundary of the small circle (of radius $1/n$).

4. Spin groups $\text{Spin}(4n + 2)$

Among the classical simple and simply connected compact Lie groups [6, 7] the spin groups $\text{Spin}(4n + 2)$ (defined as the two-sheeted universal covering groups of the special orthogonal groups $SO(4n + 2)$) are distinguished by the property of possessing a complex fundamental representation of dimension 4^n . The reason for that is that in these particular dimensions the (even) real Clifford algebras used to construct the spin groups are isomorphic to complex matrix algebras [8]. In order to analyse the mean eigenvalues:

$$X + iY = 4^{-n} \text{tr } U, \quad U \in \text{Spin}(4n + 2) \subset SU(4^n), \quad (17)$$

for $\text{Spin}(4n + 2)$ it is sufficient to consider the elements in the maximal torus of the form [8]:

$$U = (\cos \theta_1 + \gamma_1 \gamma_2 \sin \theta_1) \cdots (\cos \theta_{2n+1} + \gamma_{4n+1} \gamma_{4n+2} \sin \theta_{2n+1}). \quad (18)$$

The $4n + 2$ basis elements γ_j obey the anticommutation relations $\gamma_j\gamma_k + \gamma_k\gamma_j = -2\delta_{jk}$ and therefore generate the real Clifford algebra Cl_{4n+2} . Since only even products occur for $U \in \text{Spin}(4n + 2)$ in equation (18) we can rewrite it as

$$U = (\cos \theta_1 + \tilde{\gamma}_1 \tilde{\gamma}_2 \sin \theta_1) \cdots (\cos \theta_{2n+1} + \tilde{\gamma}_{4n+1} \sin \theta_{2n+1}), \tag{19}$$

in terms of the $4n + 1$ basis elements $\tilde{\gamma}_j = \gamma_j\gamma_{4n+2}$ which by themselves generate the real Clifford algebra Cl_{4n+1} . The complexified Clifford algebra in one dimension lower is known to be isomorphic to the algebra of complex $4^n \times 4^n$ matrices: $\mathbb{C} \otimes Cl_{4n} = \mathbb{C}(4^n \times 4^n)$. From its $4n$ generators $\tilde{\gamma}_j \in \mathbb{C}(4^n \times 4^n)$, $j = 1, \dots, 4n$, one can construct the matrix $\tilde{\gamma}_{4n+1} = i\tilde{\gamma}_1 \dots \tilde{\gamma}_{4n}$ which anticommutes with all these $4n$ generators and furthermore has the square $\tilde{\gamma}_{4n+1}^2 = -1$. When taking the trace of $U \in \text{Spin}(4n + 2)$ in this matrix representation of Cl_{4n+1} one finds readily that all traces of products of $\tilde{\gamma}$ -matrices vanish, except for $\text{tr}(\tilde{\gamma}_1 \dots \tilde{\gamma}_{4n} \tilde{\gamma}_{4n+1}) = \text{tr}(i) = 4^n i$. This leads to the following parameterization of the mean eigenvalue of a $\text{Spin}(4n + 2)$ -matrix:

$$X + iY = \prod_{j=1}^{2n+1} \cos \theta_j + i \prod_{j=1}^{2n+1} \sin \theta_j. \tag{20}$$

In order to find the boundary of the region inside the unit disc which is covered when the $2n + 1$ angles θ_j vary between 0 and 2π we search for the extremum of Y under the condition that X is kept constant. The convenient method of the Lagrange multiplier leads to the condition that all the $2n + 1$ angles θ_j have to be equal on the boundary of the trace figure of $\text{Spin}(4n + 2)$:

$$X_b + iY_b = \cos^{2n+1} \theta + i \sin^{2n+1} \theta. \tag{21}$$

With that knowledge one can also give the algebraic description

$$|X|^{2/(2n+1)} + |Y|^{2/(2n+1)} \leq 1 \tag{22}$$

of the region in the complex plane covered by the mean eigenvalues of $\text{Spin}(4n + 2)$. The grey-shaded areas in figures 2, 5, and 6 show these regions for the cases $\text{Spin}(6)$, $\text{Spin}(10)$ and $\text{Spin}(14)$, respectively. One observes cusps at the fourth roots of unity $\pm 1, \pm i$ which correspond to the discrete centre subgroup $Z(\text{Spin}(4n + 2)) = \mathbb{Z}_4$ [6]. Of course, the well-known isomorphism $\text{Spin}(6) = SU(4)$ also shows up in our analysis of mean eigenvalues. The area enclosed by the trace figure of $\text{Spin}(4n + 2)$ is readily calculated to be

$$\Omega_n = \frac{\pi(2n + 1)!}{(4^n n!)^2} \tag{23}$$

(again a rational multiple of π) and the circumference of its boundary curve (parameterized in equation (21)) is given by the integral:

$$L_n = (4n + 2) \int_0^1 dz \sqrt{z^{2n-1} + (1 - z)^{2n-1}}. \tag{24}$$

Its first four values read

$$\begin{aligned} L_1 &= 6, & L_2 &= 5 + \frac{5\sqrt{3}}{6} \ln(2 + \sqrt{3}) = 6.900\ 86, \\ L_3 &= 7.433\ 69, & L_4 &= 7.712\ 68. \end{aligned} \tag{25}$$

For large $n \rightarrow \infty$ the area approaches now quickly zero, $\Omega_\infty = 0$, whereas the circumference tends to $L_\infty = 8$. The radius of the largest circle contained in the trace figure of $\text{Spin}(4n + 2)$ is

$$R_n = 2^{-n}, \tag{26}$$

obtained by setting $\theta = \pi/4$ in equation (21).

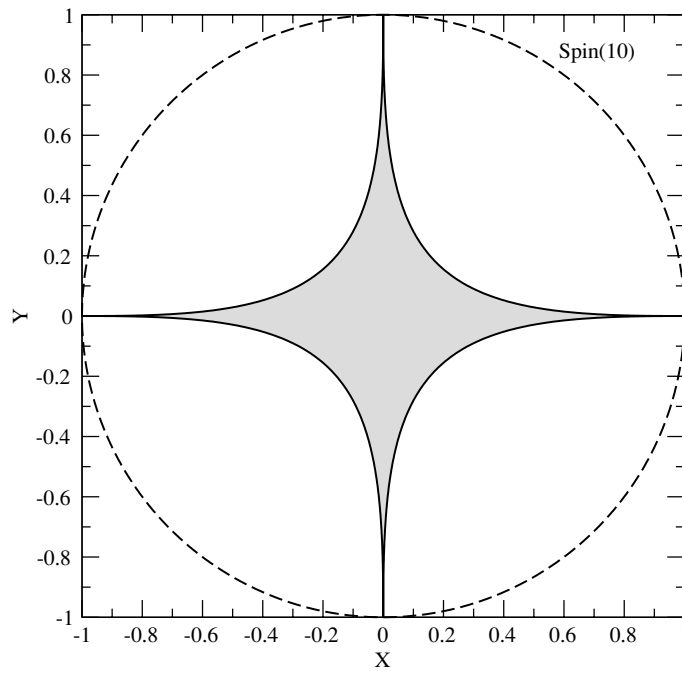


Figure 5. The trace figure of the Lie group Spin(10). The enclosed area is $15\pi/128$.

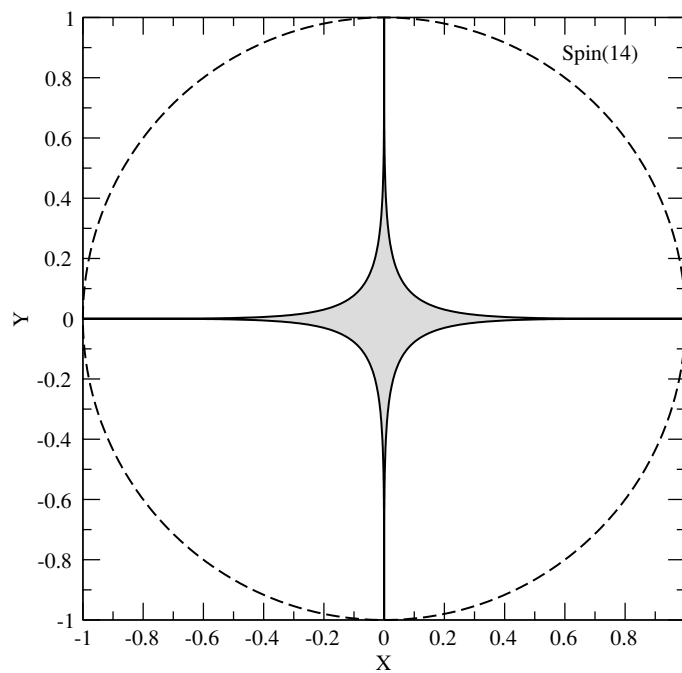


Figure 6. The trace figure of the Lie group Spin(14). The enclosed area is $35\pi/1024$.

5. Exceptional Lie group E_6

Among the five exceptional Lie groups there is exactly one candidate with a complex fundamental representation, namely E_6 with its (defining) 27-dimensional complex representation [9]. The mean eigenvalue of an E_6 -matrix is then given by the complex number:

$$X + iY = \frac{1}{27} \operatorname{tr} U, \quad U \in E_6 \subset SU(27). \tag{27}$$

Under the compact subgroup $SU(3) \times SU(3) \times SU(3)$ of E_6 this 27-dimensional complex representation decomposes as [9]

$$27 = (\mathbf{3} \otimes \bar{\mathbf{3}} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{3} \otimes \mathbf{3}) \oplus (\bar{\mathbf{3}} \otimes \mathbf{1} \otimes \bar{\mathbf{3}}), \tag{28}$$

where $\mathbf{1}$, $\mathbf{3}$ and $\bar{\mathbf{3}}$ denote the singlet, triplet and anti-triplet representations of $SU(3)$. Since $\operatorname{tr} U$ is nothing but the character of the defining 27-representation we can use this decomposition to express $\operatorname{tr} U$ in terms of products of $SU(3)$ -characters. Introducing the abbreviation $\chi(a, b) = a + b + (ab)^{-1}$ for the character of the $\mathbf{3}$ -representation of $SU(3)$ we get for the eigenvalue sum of an E_6 -matrix:

$$27(X + iY) = \chi(z_1, z_2)\chi(z_3^{-1}, z_4^{-1}) + \chi(z_3, z_4)\chi(z_5, z_6) + \chi(z_1^{-1}, z_2^{-1})\chi(z_5^{-1}, z_6^{-1}). \tag{29}$$

Here, all six complex variables $z_j = e^{i\theta_j}$ run around the unit circle. In order to get the boundary of the region covered by $X + iY$, we impose the zero-derivative conditions $\partial(X + iY)/\partial z_{1,3,5} = 0$. These allow us in a first step to eliminate half of the variables: $z_2 = z_1^{-2}$, $z_4 = z_3^{-2}$, $z_6 = z_5^{-2}$. Any further zero-derivative condition on the reduced expression for $X + iY$ requires one of the three remaining variables to be a cube root of unity. We set $z_3 = 1$, and get an expression which is symmetric under the exchange $z_1 \leftrightarrow z_5$. Setting finally $z_1 = z_5 = \xi \pm i\sqrt{1 - \xi^2}$ with $-1 \leq \xi \leq 1$ we find the following parameterization of the boundary curve¹:

$$X_b = \frac{1}{27}(8\xi^4 + 12\xi^2 + 16\xi - 9), \quad Y_b = \pm \frac{8}{27}(1 - \xi)^2(2 + \xi)\sqrt{1 - \xi^2}, \tag{30}$$

where the signs \pm correspond to the branches in the upper/lower halfplane. As an example, the parameter value $\xi = -1$ gives the boundary point $X_b = -5/27$, $Y_b = 0$ on the negative real axis. Figure 7 shows the trace figure of the exceptional Lie group E_6 . Besides the dihedral symmetry D_3 one observes again cusps at the cube roots of unity 1 , $(-1 \pm i\sqrt{3})/2$ which reflect the centre subgroup of $E_6 : Z(E_6) = \mathbb{Z}_3$ [6]. The area of the grey-shaded region in figure 7 amounts to

$$\Omega(E_6) = \frac{20\pi}{243} \tag{31}$$

(again a rational multiple of π) and the circumference of the boundary curve is given by the (elliptic) integral:

$$L(E_6) = \frac{4}{9} \int_1^4 ds(s - 1)\sqrt{5(s - 3)^2 + 16s^{-1}} = 5.59601. \tag{32}$$

The radius of the largest circle contained in the trace figure of E_6 is

$$R(E_6) = \frac{5}{27}. \tag{33}$$

This largest circle meets the three boundary points: $5(1 \pm i\sqrt{3})/54$, ($\xi = 1/2$) and $-5/27$, ($\xi = -1$).

¹ Although some steps in this derivation may seem to be not perfectly rigorous the main result for the boundary curve equation (30) has been confirmed by detailed numerical investigations.

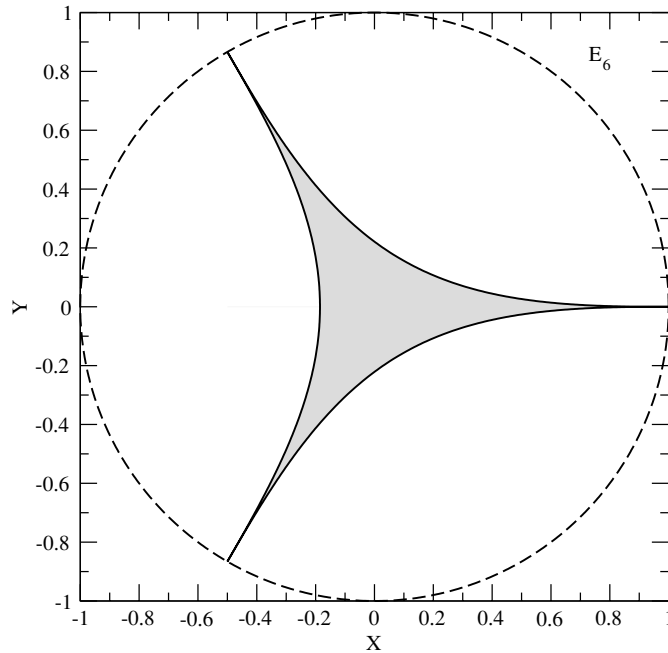


Figure 7. The trace figure of the exceptional Lie group E_6 . The enclosed area is $20\pi/243$.

All the other simple and simply connected compact Lie groups G have only selfconjugate representations [7]. Therefore their mean eigenvalues $X(G)$ are real and confined to the interval $[-1, 1]$. In the cases $G = SU(2), Sp(n)$ (the symplectic groups), $Spin(2n + 1)$ and E_7 with centre subgroup $Z(G) = \mathbb{Z}_2$ [6] of order 2, it follows immediately from continuity that the whole interval $[-1, 1]$ will be covered by the mean eigenvalues: $-1 \leq X(G) \leq 1$. The same feature applies also to $Spin(4n)$ with centre $Z(Spin(4n)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ [6]. The lower limit $X(G) = -1$ is in each case reached by the negative unit matrix contained in the centre of these simple and simply connected compact Lie groups. This leaves the exceptional Lie groups G_2, F_4 and E_8 with trivial centre [6] as special cases since for them the (trivial) lower bound $X(G) = -1$ may not be reached. In the next section we determine the proper lower bounds on the mean eigenvalues for these three exceptional groups.

6. Exceptional Lie groups G_2, F_4 and E_8 with trivial centre

We start with the exceptional Lie group G_2 of rank 2. Its lowest dimensional nontrivial representation is the real 7-representation (i.e., the defining representation of G_2) and it decomposes under the subgroup $SU(3) \subset G_2$ as [9]

$$7 = 1 \oplus 3 \oplus \bar{3}. \tag{34}$$

This property allows us to write the mean eigenvalue of a G_2 -matrix:

$$X = \frac{1}{7} \text{tr } U, \quad U \in G_2 \subset SO(7), \tag{35}$$

in terms of the $SU(3)$ -character as follows:

$$7X = 1 + \chi(z_1, z_2) + \chi(z_1^{-1}, z_2^{-1}) = 1 + 2 \text{Re } \chi(z_1, z_2). \tag{36}$$

From our analysis of $SU(3)$ in section 2 we can take over the inequality: $-3/2 \leq \text{Re } \chi(z_1, z_2) \leq 3$ (see also figure 1) and deduce from it that the allowed range for the mean eigenvalues of G_2 -matrices is

$$-\frac{2}{7} \leq X(G_2) \leq 1. \tag{37}$$

We continue with the exceptional Lie group F_4 of rank 4. Its lowest dimensional nontrivial representation is the real **26**-representation (to be viewed as the defining representation of F_4). It decomposes under the subgroup $SU(3) \times SU(3) \subset F_4$ as [9]

$$\mathbf{26} = (\mathbf{8} \otimes \mathbf{1}) \oplus (\mathbf{3} \otimes \mathbf{3}) \oplus (\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}), \tag{38}$$

with **8** being the (real) octet representation of $SU(3)$. This fact allows us again to write the mean eigenvalue of a F_4 -matrix:

$$X = \frac{1}{26} \text{tr } U, \quad U \in F_4 \subset SO(26), \tag{39}$$

in terms of $SU(3)$ -characters as

$$26X = \chi(z_1, z_2)\chi(z_1^{-1}, z_2^{-1}) - 1 + \chi(z_1, z_2)\chi(z_3, z_4) + \chi(z_1^{-1}, z_2^{-1})\chi(z_3^{-1}, z_4^{-1}), \tag{40}$$

with z_j four complex variables running on the unit circle. By imposing the zero-derivative conditions $\partial X / \partial z_{1,3} = 0$ we can eliminate two variables: $z_2 = z_1^{-2}, z_4 = z_3^{-2}$. Then we redefine $z_3 = z_2/z_1$ in the reduced expression and the further zero-derivative condition $\partial X / \partial z_1 = 0$ fixes $z_2 = z_1^3$. Finally, we set $z_1 = e^{i\theta}$ and get the quadratic expression

$$13X = 2(2 + \cos 3\theta)^2 - 5 \geq -3, \tag{41}$$

from which one can easily read off the lower bound. Altogether this implies that the mean eigenvalues of F_4 -matrices lie in the interval

$$-\frac{3}{13} \leq X(F_4) \leq 1. \tag{42}$$

The lower limit value $-3/13$ has also been confirmed in numerical studies starting from equation (40).

The most demanding case is that of the exceptional Lie group E_8 of rank 8. The lowest dimensional nontrivial representation of E_8 is the real **248**-representation, i.e., the adjoint representation on its own Lie algebra. It decomposes under the subgroup $E_6 \times SU(3) \subset E_8$ as [9]

$$\mathbf{248} = (\mathbf{78} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{8}) \oplus (\mathbf{27} \otimes \mathbf{3}) \oplus (\bar{\mathbf{27}} \otimes \bar{\mathbf{3}}), \tag{43}$$

where the adjoint **78**-representation of E_6 decomposes under $SU(3) \times SU(3) \times SU(3)$ as

$$\mathbf{78} = (\mathbf{8} \otimes \mathbf{1} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{8} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{8}) \oplus (\mathbf{3} \otimes \mathbf{3} \otimes \bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \otimes \mathbf{3}). \tag{44}$$

The mean eigenvalue of an E_8 -matrix:

$$X = \frac{1}{248} \text{tr } U, \quad U \in E_8 \subset SO(248), \tag{45}$$

is proportional to the character of the **248**-representation and the decompositions equations (28), (43), (44) give us $248X$ as a Laurent-polynomial in eight (unitary) variables z_1, \dots, z_8 with integer coefficients. Setting all these eight variables equal to each other: $z_j = \xi \pm i\sqrt{1 - \xi^2}, j = 1, \dots, 8$, with $-1 \leq \xi \leq 1$, we get the expression:

$$31X = 8\xi^6 + 24\xi^5 + 12\xi^4 - 8\xi^3 - 6\xi^2 + 1. \tag{46}$$

The sixth degree polynomial on the right-hand side of equation (46) is shown in figure 8. One sees that within the interval $-1 \leq \xi \leq 1$ its absolute minimum value -1 is reached on the left boundary $\xi = -1$. Having found this lower bound we can conclude that the allowed range for the mean eigenvalues of E_8 is

$$-\frac{1}{31} \leq X(E_8) \leq 1. \tag{47}$$

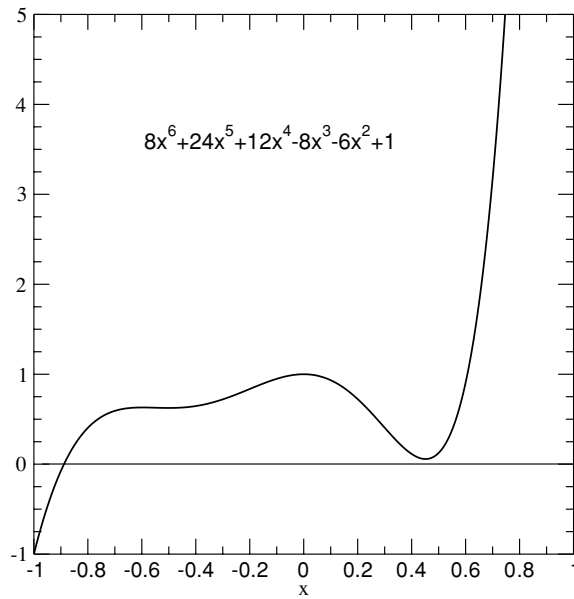


Figure 8. The polynomial $8\xi^6 + 24\xi^5 + 12\xi^4 - 8\xi^3 - 6\xi^2 + 1$ of degree six in the interval $-1 \leq \xi \leq 1$. When multiplied with a factor $1/31$ it determines the range of the real mean eigenvalues for the exceptional Lie group E_8 in the fundamental **248**-representation.

We note that one can equivalently eliminate half of the eight variables through zero-derivative conditions as $z_j = z_{j-1}^{-2}$, $j = 2, 4, 6, 8$, and then go with the remaining four variables onto the diagonal: $z_1 = z_3 = z_5 = z_7$. This procedure leads to the same expression as in equation (46). The lower limit value $-1/31$ has again been confirmed in numerical studies starting from the full eight-parameter form of $X(E_8)$.

This concludes our analysis of the mean eigenvalues for the exceptional Lie groups G_2 , F_4 and E_8 with trivial centre. We have derived the rational, negative lower bounds: $-2/7$, $-3/13$ and $-1/31$, respectively.

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